if $D = f_{xx} f_{yy} - (f_{xy})^2 = 0$

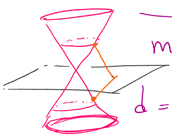
$\Rightarrow$ we know nothing.

Ex $f(x,y) = y^2 \Rightarrow D = 0$ at $(x,y) = 0$

but this is a local minimum

Ex

find the closest distance from the surface $z^2 = x^2 + y^2$ to $(4,2,0)$



minimize distance to $(4,2,0)$

$d = (x-4)^2 + (y-2)^2 + z^2$

(could do $\sqrt{(x-4)^2 + (y-2)^2 + z^2}$ but bad computations)

Constraint is $z^2 = x^2 + y^2$ $\quad (g = x^2 + y^2 - z^2)$

Sub into d

$d = (x-4)^2 + (y-2)^2 + x^2 + y^2$

$= x^2 - 8x + 16 + y^2 - 4y + 4 + x^2 + y^2$

$= 2x^2 + 2y^2 - 8x - 4y + 20$

$d_x = 4x - 8$
$d_y = 4y - 4$ $\Big\}$ zero if $x = 2$, $y = 1$

$z^2 = x^2 + y^2 = 5$

$\Rightarrow (2, 1, \sqrt{5})$
$(2, 1, -\sqrt{5})$

$\Rightarrow$ Closest distance

$= \sqrt{(2-4)^2 + (1-2)^2 + (0-\sqrt{5})^2}$

$= \sqrt{4 + 1 + 5} = \sqrt{10}$

Lagrange Multipliers

To maximize (or min) $f(x,y,z)$ subject to constraint $g(x,y,z) = c$

$\Rightarrow$ it is all $(x,y,z), \lambda$ s.t.

$\nabla f = \lambda \nabla g$ $\qquad \nabla g \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Can $\lambda = 0$?

Ex. $f(x,y) = x^2 + y^2$

$g = y - x^2 = 0$

Ex. $f(x,y) = x^2 + y^2$

$g = y - x^2 = 0$

Ex Find critical points of $f(x,y) = x^2 + y^2$ with constraint $xy = 1$

if $g = 0 \Rightarrow xy = 0$

$= \frac{1}{y}$

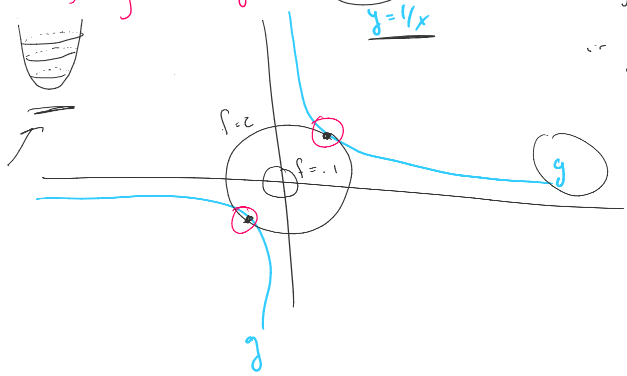
$\Rightarrow y^4 = 1 \Rightarrow y = \pm 1$

$z = f(x,y) = x^2 + y^2$ $\quad xy = 1$

$y = 1/x$

Critical @ $x = 1$, $y = 1$

or $x = -1$, $y = -1$



$$\begin{cases} 2x = \lambda y & \text{— mult by } y \quad 2 = 2xy = \lambda y^2 \\ 2y = \lambda x & \leftarrow \text{by } x \quad 2 = 2yx = \lambda x^2 \\ xy = 1 \end{cases}$$

Disc 107

In Lagrange multipliers, what does $\lambda = 0$ mean?

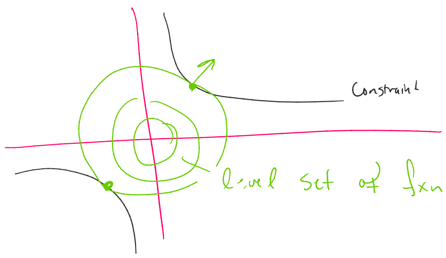
To max or min f w/ constraint $g = c$

we solve $\left\{ \begin{array}{l} \nabla f = \lambda \nabla g \\ g = c \end{array} \right\} \leftarrow$ in 3D (x,y,z) 4 equations 4 unknowns

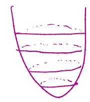
with $\nabla g = \langle 0,0,0 \rangle$

Can $\lambda = 0$?

ie if $\lambda = 0$, can we get a min or max?



Ex $f(x,y) = x^2 + y^2$



$g = y - x^2 = 0$

method 1

$y = x^2$ Sub into f $\Rightarrow$ $f = x^2 + (x^2)^2$
$= x^2 + x^4$

$f' = 2x + 4x^3 = 0$

$x = 0$

or $2 + 4x^2 = 0$

$2x^2 = -1$

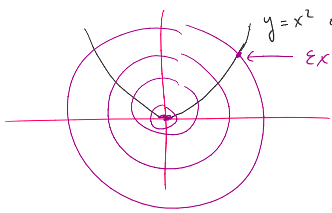
$x^2 = -1/2$ $\Longrightarrow$ no solution

$\Rightarrow$ $x = 0$ $y = x^2 = 0$

$f = 0$ @ critical point

$y - x^2 = 0$

min or max? min



Method 2 (Lagrange)

$f = x^2 + y^2$ $\nabla f = \langle 2x, 2y \rangle$

$g = y - x^2 = 0$ $\nabla g = \langle -2x, 1 \rangle$ $\underbrace{\nabla f = \lambda \nabla g}$

$1 = -\lambda$

① $2x = -2x\lambda$
② $2y = \lambda$
③ $y - x^2 = 0$

$x \neq 0$ $\lambda = -1$

② $2y = -1$ $y = -1/2$

③ $-\frac{1}{2} = x^2$

x is imaginary contradiction

$\Rightarrow$ $x = 0$

① $2x = -2x(2y)$

$x = -2xy$

③ $x = -2x(x^2)$

$x = -2x^3$

$x = 0$ $\Longrightarrow$ $y = 0$ $\Rightarrow$ $\lambda = 0$

$x = -2x^3$

$x=0 \Rightarrow y=0 \Rightarrow \lambda = 0$

or $1 = -2x^2 \Rightarrow x$ imaginary

1) find closest distance from $z^2=x^2+y^2$ to $(4,2,0)$

Could minimize $d=\sqrt{(x-4)^2+(y-2)^2+z^2}$

Equivalent to minimize $d=(x-4)^2+(y-2)^2+z^2$

2) max $f=x^2+y^2$ w/ constraint $xy=1$

$g=xy-1$

$y=0$ cannot work! b/c $xy=1$

$$\begin{cases} ① & 2x=\lambda y \\ ② & 2y=\lambda x \\ ③ & xy=1 \end{cases}$$

$$\left.\begin{aligned} \frac{2x}{y} &= \lambda \\ \frac{2y}{x} &= \lambda \end{aligned}\right\} \longrightarrow \frac{2x}{y}=\frac{2y}{x}$$

$2x^2=2y^2$

$x^2=y^2$

$x=y \rightarrow$ ③ $x^2=1 \Rightarrow x=\pm 1$

or $x=-y$ impossible by ③ b/c $\cancel{1=xy=x(-x)}$
$\cancel{-x^2=1}$

Get $(x,y)=$ ((…

③

3) max $g(x,y)=xy-x+y$ in rectangle
$-2\le x\le 2$
$-2\le y\le 2$

1) find closest distance from $z^2=x^2+y^2$ to $(4,2,0)$

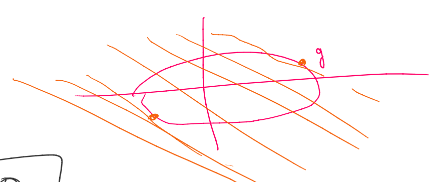
$f=(x-4)^2+(y-2)^2+z^2$

$g=x^2+y^2-z^2$

$\nabla f = \lambda \nabla g$

$f=3x+2y+4z$

$g=x^2+2y^2+6z^2=25$



$\boxed{x\neq 0}$

$$\begin{cases} ① & \boxed{3=\lambda\, 2x} \\ ② & 2=\lambda\, 4y \end{cases}$$

$\lambda = \frac{3}{2x}$

② $\Rightarrow 2=\left(\frac{3}{2x}\right)4y=\frac{6y}{x}$

$\frac{x}{3}=\frac{2x}{6}=y$

$(,1)$

$-1,-1)$

$$\begin{cases} 3 = \lambda\, 2x & (1)\\ 2 = \lambda\, 4y & (2)\\ 4 = 12\lambda z & (3)\\ x^2+2y^2+6z^2=25 & (4)\end{cases}$$

$\lambda = \frac{3}{2x}$

(2) $\Rightarrow 2 = \left(\frac{3}{2x}\right)4y = \frac{6y}{x}$ $\qquad \frac{x}{3} = \frac{2x}{6} = y$

(3) $\Rightarrow 4 = 12z\left(\frac{3}{2x}\right) = \frac{18z}{x}$ $\qquad \frac{2x}{9} = \frac{4x}{18} = z$

(4) $\Rightarrow x^2 + 2\left(\frac{x}{3}\right)^2 + 6\left(\frac{2x}{9}\right)^2 = 25$

max $\rightarrow \left(\alpha, \frac{\alpha}{3}, \frac{2\alpha}{9}\right)$

min $\rightarrow \left(-\alpha, \frac{-\alpha}{3}, -\frac{2\alpha}{9}\right.$

(1) $x = \frac{3}{2\lambda}$, $y = \frac{2}{4\lambda}$, $z = \frac{4}{12\lambda}$

Plug into (4)

$\left(\frac{3}{2\lambda}\right)^2 + 2\left(\frac{2}{4\lambda}\right)^2 + \left(\frac{4}{12\lambda}\right)^2 = 25$

Solve for λ

then get x, y, z

$$x^2 + \frac{2x^2}{9} + \frac{6\cdot 16x^2}{(18)^2} = 25$$

$$x^2\left(1 + \frac{2}{9} + \frac{6\cdot 16}{18^2}\right) = 25$$

$$x = \pm\sqrt{\frac{25}{\left(1+\frac{2}{9}+\frac{6\cdot 16}{18}\right)}} = \pm\alpha$$

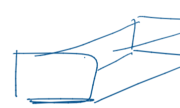
$\frac{6\cdot 16}{18^2} =$

$= 1$

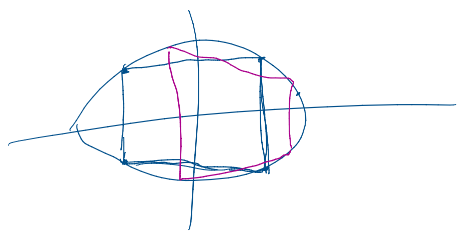
$= 27$

$x^2 =$

$x =$



in 2D



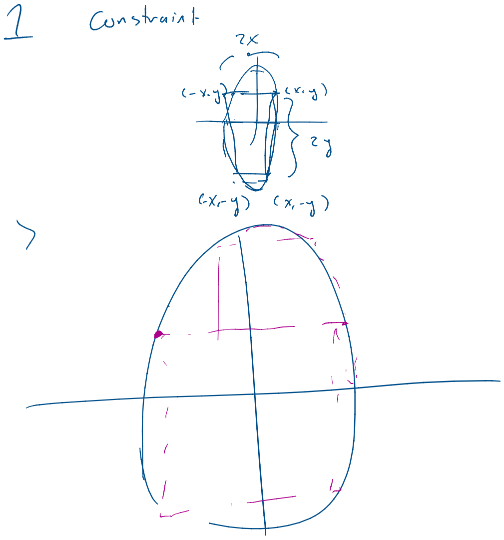
Ex $g = \frac{x^2}{9} + \frac{y^2}{64} = 1$ Constraint

$f = 4xy$

$\nabla f = \langle 4y, 4x\rangle$

$\nabla g = \left\langle \frac{2x}{9}, \frac{2y}{64}\right\rangle$

$\nabla f = \lambda \nabla g$



2 Constraints

$$\left.\begin{aligned} g &= x+2y+2z = 20\\ h &= x^2+y^2-z = 0\\ f &= x^2+y^2+z^2 \end{aligned}\right\}$$

max & min

$\nabla f = \lambda\nabla g + \mu\nabla h$ $\qquad$ Solve for x, y, z, μ, λ

$\nabla f = \langle 2x, 2y, 2z\rangle$

$\nabla g = \langle 1, 2, 2\rangle$

$\nabla h = \langle 2x, 2y, -1\rangle$

Izak Oltman at 3/13/2020 1:41 PM

$x)$

$$\frac{2 \cdot 3 \cdot 2^4}{2^2 \cdot 3^4} = \frac{2 \cdot 2^2}{3^3}$$

$$= \frac{8}{27}$$

$$+ \frac{2}{9} + \frac{8}{27}$$

$$\frac{+6+8}{27} = \frac{41}{27}$$

$$\frac{27 \cdot 25}{41}$$

$$\pm 5 \sqrt{\frac{27}{41}}$$

$\nabla g = \langle 1, 2, 2 \rangle$

Izak Oltman at 3/13/2020 1:41 PM

$\nabla h = \langle 2x, 2y, -1 \rangle$

① $2x = \lambda + \underline{2x\mu}_{2x}$

② $2y = 2\lambda + \underline{2y\mu}_{2y}$

③ $2z = 2\lambda - \underline{\mu}_{-1}$

④ $x + 2y + 2z = 20$

⑤ $x^2 + y^2 - z = 0$

if $\mu = 1$

① $2x(1-\mu) = \lambda$

② $y(1-\mu) = \lambda$

$2x(1-\mu) = y(1-\mu)$

if $\mu \neq 1$

$\Rightarrow 2x = y$

④ $\Rightarrow x + 2(2x) + 2z = 20$

$5x + 2z = 20$

$z = 10 - \frac{5}{2}x$

⑤ $\Rightarrow$ $x^2 + 4x^2 - 10 + \frac{5}{2}x = 0$

$5x^2 + \frac{5}{2}x - 10 = 0$

$x = \dfrac{-\frac{5}{2} \pm \sqrt{\frac{5^2}{2^2} + 200}}{10}$